ELSEVIER Journal of Pure and Applied Algebra 112 (1996) 207–217

JOURNAL OF PURE AND APPLIED ALGEBRA

The Brown–Peterson $[2^k]$-series revisited

David Copeland Johnson [a,*], George Nakos [b,1]

[a] University of Kentucky Lexington, KY 40506, USA
[b] The Johns Hopkins University, Baltimore, MD 21218, USA

Communicated by J.D. Stasheff; received 17 March 1995; revised 13 September 1995

Abstract

Each coefficient $a_{k,s}$ of the $[2^k]$-series for Brown–Peterson homology has a distinguished shortest monomial which is determined by the dyadic expansion of $s+1$.

1991 Math. Subj. Class.: 55N22

1. Introduction

A Brown–Peterson spectrum BP has a formal group law $F(X,Y)$ which is a formal power series in the two indeterminates X and Y over its coefficient ring. The eponymous $[2^k]$-series arises as follows. We start with $[1]X$ defined to be $X = F(X,0) = F(0,X)$. For any positive integer m ($m = 2^k$, for example), $[m]X$ is defined to be $F([m-1]X,X)$. It is more illuminating to define $[2]X = F(X,X)$ and to use the fact that $[mn]X = [m]([n]X)$ to define $[2^k]X = [2]([2^{k-1}]X)$ by induction. (See the proof of Lemma 11.) Our interest is in the Brown–Peterson spectrum for the prime 2, with the $[2^k]$-series,

$$[2^k] = \sum_{0\le s} a_{k,s}X^{s+1}, \quad a_{k,s} \in BP_{2s}, \tag{1}$$

and with the coefficients $a_{k,s}$. The coefficients of the $[2^k]$-series generate the relations in the Brown–Peterson homology of $B\mathbf{Z}/2^k$, the classifying space of the cyclic group $\mathbf{Z}/2^k$. When $k=1$ and $s = 2^n - 1$, these generate the Brown–Peterson coefficient ring, BP_*.

* Corresponding author. E-mail: johnson@ms.uky.edu.

[1] This research was conducted when this author was visiting the Johns Hopkins University while on sabbatical from the U.S. Naval Academy, Annapolis, MD 21402, USA. We wish to thank both institutions for allowing this to happen.

SSDI 0022-4049(95)00140-9

Recall that the Brown–Peterson coefficient ring BP_* is isomorphic to the ring of polynomials $\mathbf{Z}_{(2)}[v_1, v_2, \ldots]$ with generators v_n of dimension $2^{n+1} - 2$. By tradition v_0 is taken to be the integer 2. We need the ideal $I_\infty = (2, v_1, v_2, \ldots)$. We say that a homogeneous element a (a monomial, for example) of BP_* is of length at least k provided that $a \in (I_\infty)^k$. Any element $a \in BP_*$ has a unique expression as $a = \sum_I a_I v^I$, where I is a sequence of nonnegative integers (all but finitely many of which are zero) $I = (i_1, \ldots, i_n, \ldots)$, $v^I = v_1^{i_1} \cdots v_n^{i_n} \cdots$, $a_I \in \mathbf{Z}_{(2)}$, all but finitely many a_I are zero. If an element $a \in BP_*$ is of length exactly k and if $a = \sum_I a_I v^I$ as above, then each nonzero $a_I v^I$ has length at least k and *some* nonzero $a_J v^J$ has length exactly k. We say that $a_J v^J$ is a shortest monomial of a.

These $[2^k]$-series coefficients $a_{k,s}$ are important and they can be *known*: for example, $a_{3,10}$ is just the sum of six monomials decorated with integers — integers in the billions and trillions. What we seek is to *understand* the $a_{k,s}$. This paper specifies a distinguished shortest monomial in each $a_{k,s}$, described simply in terms of the dyadic expansion of the integer $s + 1$.

Theorem 1. *Let $a_{k,s}$ be the dimension $2s$ coefficient of the Brown–Peterson $[2^k]$-series. Let the integer $s + 1$ have the dyadic expansion*

$$s + 1 = e_1 + e_2(2^1) + e_3(2^2) + \cdots + e_t(2^{t-1}), \quad \text{where } 0 \le e_i < 2. \tag{2}$$

Let $\alpha = e_1 + e_2 + \cdots + e_t$ be the number of 1's in the dyadic expansion of $s + 1$. Then $a_{k,s} \in I_\infty^{(k+1)\alpha - 1}$ and

$$a_{k,s} \equiv 2^{(k-1)\alpha} v_1^{\alpha-1} 2^{e_1} v_1^{e_2} \cdots v_{t-1}^{e_t} \quad \text{modulo } (2^{(k-1)\alpha+2}) \cap I_\infty^{(k+1)\alpha-1} + I_\infty^{(k+1)\alpha}. \tag{3}$$

Section 4 gives examples of some $a_{k,s}$ and their shortest monomials as specified by this theorem. Here are some of its easy implications.

Corollary 2. *When $k = 1$, the integer 4 never divides the term $a_{1,s}$.*

Corollary 3. *If $k = 1$ and $s = 2r + 1$ is odd, then the integer 2 fails to divide the term $a_{1,2r+1}$.*

Corollary 4. *Unless $s + 1$ is a power of 2, the generator v_1 divides the shortest monomial of $a_{k,s}$.*

The proof in the simple case when $k = 1$ goes as follows. Let α and the e_i be as in the statement of the theorem. Modulo $(4) + I_\infty^{2\alpha}$, $a_{1,s}$ is a sum of certain products of the form $v_1 a_{1,h} a_{1,i}$ where $h + i + 1 = s$. By an induction, both $a_{1,h}$ and $a_{1,i}$ obey the theorem; so each product is the monomial

$$v_1^{\alpha-1} 2^{e_1} v_1^{e_2} \ldots v_{t-1}^{e_t} \quad \text{modulo } (4) + I_\infty^{2\alpha}. \tag{4}$$

So Section 2 shows that modulo $(4) + I_\infty^{2\alpha}$, $a_{1,s}$ is a sum of the desired monomials. How many monomials are in this sum, an odd number or an even one? In Section 3, we count and our answer is in terms of a Stirling number of the second kind; it is odd.

The proof of our main result is developed in Sections 2 and 3. Section 2 is a reworking of certain lemmas from [3] and [6]. Those papers and Section 2 are valid for all primes. Section 3 completes the proof with counting arguments specific for $p = 2$. We also state the $p = 3$ analog of the main result without proof. Section 4 gathers examples of applications of Theorem 1.

This paper revisits the subject of the papers [3] and [6]. Here is how this joint paper germinated. After a reading of [3], Andy Baker asked: does the integer 2 divide the term $a_{1,s}$ only when s is even? (The answer is yes.) That question stimulated the work behind preprint [4]. This preprint, in turn, stimulated the research described in the preprint [7]. Jim Stasheff had a central role in the idea that the highlights of the two preprints [4] and [7] should appear together in one paper. We are grateful to Andy Baker and to Jim Stasheff for their respective roles in the development of this note. We thank the referee for his prompt, thorough, and helpful reading of our typescript.

2. The short monomial

Now let p be a prime – 2 or odd. The coefficient ring for the Brown–Peterson theory at p, BP_*, is isomorphic to $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ with the degree of v_n being $2(p^n - 1)$. We have a complete list of prime ideals of BP_* which are invariant under the action of BP-operations: $I_0 = (0), I_1 = (p), I_2 = (p, v_1), \ldots, I_n = (p, v_1, \ldots, v_{n-1}), \ldots,$ and $I_\infty = (p, v_1, \ldots, v_{n-1}, \ldots) = \bigcup_{n=0}^{\infty} I_n$. The $[p^k]$-series is constructed inductively just as was the $[2^k]$-series. For this section, we write

$$[p^k]X = \sum_{0 \le s} a_{k,s} X^{s+1}, \quad a_{k,s} \in BP_{2s}. \tag{5}$$

The major result of [6] is the following.

Proposition 5. *Let $a_{k,s}$ be the dimension $2s$ coefficient of the Brown–Peterson $[p^k]$-series. Let the positive integer $s+1$ have the p-adic expansion*

$$s + 1 = e_1(1) + e_2(p^1) + e_3(p^2) + \cdots + e_t(p^{t-1}), \quad 0 \le e_i < p. \tag{6}$$

Let f be defined by $(p-1)f + 1 = e_1 + e_2 + \cdots + e_t$. Then $a_{k,s} \in I_1^{ke_1} I_2^{ke_2} \cdots I_t^{ke_t} I_\infty^f$.

Since we are computing with the p-adic expansions of several integers simultaneously, let us introduce the following notation. Any positive integer $t+1$ has a p-adic expansion

$$t + 1 = \sum_{0 < i} (t+1)_i p^{i-1}, \quad 0 \le (t+1)_i < p. \tag{7}$$

Define $v_p(t+1) = j$ if $t+1 = cp^j$, where $\gcd(c,p) = 1$. (Thus $(t+1)_{j+1}$ is not zero and $(t+1)_i = 0$ for $i \le j$.) Using the p-adic expansion (7), define $\alpha_p(t+1)$ by

$$\alpha_p(t+1) = \sum_{0<i} (t+1)_i. \tag{8}$$

Recall that BP_* is "sparse": it is concentrated in nonnegative dimensions which are multiples of $2p-2$. Thus, $a_{k,s} = 0$ unless $s \equiv 0$ modulo $(p-1)$. Consequently, we may always assume that s is a multiple of $p-1$. For such an $s+1$, the following equation:

$$(p-1)g = [k(p-1)+1]\alpha_p(s+1) - 1 \tag{9}$$

defines an integer g.

Corollary 6. *For the prime p, the exponent k, and the positive integer $s+1$, let g be as in (9). Then $a_{k,s} \in I^g_\infty$. (Note that $g = (k+1)\alpha_2(s+1) - 1$ when $p = 2$.)*

Lemma 7. *If $s+1 = \sum_{0\le t\le w} u_t(t+1)$, where $0 \le u_t$ and $0 < t+1$, then*

$$\alpha_p(s+1) \le \sum_{0\le t\le w} \alpha_p(u_t)\alpha_p(t+1). \tag{10}$$

Proof. Lemma 3.1 (i, ii) of [3]. □

Definition 8. The sum $s+1 = \sum_{0\le t\le w} u_t(t+1)$, where $0 < t+1$ and $0 \le u_t < p$ is subordinate to the p-adic expansion of $s+1$ provided that for each $0 < i$,

$$(s+1)_i = \sum_{0\le t\le w} k_t(t+1)_i. \tag{11}$$

Lemma 9. *Suppose $s+1 = \sum_{0\le t\le w} u_t(t+1)$, where $0 < t+1$ and $0 \le u_t < p$. Then*

$$\alpha_p(s+1) = \sum_{0\le t\le w} \alpha_p(u_t)\alpha_p(t+1) = \sum_{0\le t\le w} u_t\alpha_p(t+1) \tag{12}$$

if and only if the sum for $s+1$ is subordinate to the p-adic expansion of $s+1$.

Proof. The "if" part is obvious. The "only if" part follows by iteration of the following special case. □

Lemma 10. *Suppose $a = b+c$, $a = a_1 + pa_0, b = b_1 + pb_0, c = c_1 + pc_0$ with $0 \le a-1, b_1, c_1 < p$. If $\alpha_p(a) = \alpha_p(b) + \alpha_p(c)$, then $a_1 = b_1 + c_1$.*

Proof. Note that $\alpha_p(a) = \alpha_p(a_1) + \alpha_p(pa_0) = a_1 + \alpha_p(a_0)$ and similarly for the sums representing b and c. Since $a = a_1 + pa_0 = b_1 + pb_0 + c_1 + pc_0 = b_1 + c_1 + pb_0 + pc_0$, we have

$$\alpha_p(a) \leq \alpha_p(b_1 + c_1) + \alpha_p(b_0) + \alpha_p(c_0) \leq b_1 + c_1 + \alpha_p(b_0) + \alpha_p(c_0) \tag{13}$$

$$= \alpha_p(b) + \alpha_p(c) = \alpha_p(a). \tag{14}$$

Thus all the inequalities of (13) and (14) are equalities and $\alpha_p(b_1 + c_1) = b_1 + c_1$; this number is less than p. Modulo p, $a_1 \equiv b_1 + c_1; a_1 < p$. Thus, $a_1 = b_1 + c_1$. □

Suppose $U = (u_0, u_1, \ldots)$ is a sequence of non-negative integers such that u_t is nonzero for only finitely many t. Let $a_k^U = a_{k,0}^{u_0} a_{k,1}^{u_1} \ldots$ and $v^U = v_0^{u_0} v_1^{u_1} \ldots$. If $u_0 + u_1 + \ldots = \sum_{0 \leq t} u_t = p^i$, let $\binom{p^i}{U}$ be the multinomial coefficient $\binom{p^i}{u_0, u_1, \ldots}$.

Recall from [8] that there are homology classes $m_{p^i - 1} \in H_{2p^i - 2}(BP; \mathbf{Q})$ (which contains $BP_{2p^i - 2}$) such that

$$v_n = pm_{p^n - 1} - \sum_{0 \leq j < n} m_{p^j - 1}(v_{n-j})^{p^j}. \tag{15}$$

Thus,

$$p^i m_{p^i - 1} \in I_\infty^i \subset BP_* \tag{16}$$

and

$$p^2 m_{p^2 - 1} = pv_2 + v_1^{p+1}. \tag{17}$$

The Brown–Peterson theory has a logarithm defined by

$$\log X = \sum_{0 \leq i} m_{p^i - 1} X^{p^i} \tag{18}$$

such that

$$\log([p]X) = p \log X. \tag{19}$$

Lemma 11. *For $k > 0$, the power series $[p^k]X$ is an element of the ideal $[(p^{(k-1)}) \bigcap I_\infty^k + I_\infty^{k+1}]BP_*[[X]]$ of $BP_*[[X]]$. Modulo $I_\infty^{k+1} BP_*[[X]]$, $[p^k]X = p^{k-1}[p]X = \sum_{0 \leq n} p^{k-1} a_{1,p^n - 1} X^{p^n}$.*

Proof. Since (p^0) is the entire ring, the $k = 1$ case for the first statement is trivial. The second statement is well known for $k = 1$. Now $[p^k]X = [p]([p^{k-1}]X) = p[p^{k-1}]X + a_{1,p-1}([p^{k-1}]X)^p + \cdots$. So the lemma follows by induction on k. □

We now have the notations and preliminaries necessary to state and prove the key proposition of this section.

Proposition 12. *Fix the prime p and the exponent k. Let*

$$[p^k]X = \sum_{0 \leq s} a_{k,s} X^{s+1}, \quad a_{k,s} \in BP_{2s}. \tag{20}$$

Let $s+1$ be a positive integer which is not a power of p and such that s is a multiple of $p-1$. Let $g = g(p,k,s+1)$ as in (9). *Then*

$$a_{k,s} \equiv -\sum_{U,0\le i} m_{p^i-1}\binom{p^i}{U} a_k^U \quad \text{modulo } I_\infty^{g+1} \tag{21}$$

where the sum (21) *is taken over all sequences of non-negative integers $U = (u_0, u_1, \ldots)$ satisfying* (22)–(25):

$$s+1 = \sum_t u_t(t+1), \quad 0 < t+1, \tag{22}$$

$$p^i = \sum_t u_t, \tag{23}$$

$$0 \le u_t < p. \tag{24}$$

The sum in (22) *is subordinate to the p-adic expansion of $s+1$.* (25)

Remark 13. Since $0 \le u_t < p, v_p\left(\binom{p^i}{U}\right) = v_p(p^i!)$. So $\binom{p^i}{U} m_{p^i-1} \in BP_*$ and the expression in (21) is a formula in BP_*. By Corollary 2, $a_{k,s} \in I_\infty^g$; so our terms are on the brink of being in I_∞^{g+1}.

Proof of Proposition 12. Choose a large q so that all $p^q m_{p^i-1} \in BP_*$ in (21). By (18) and (19),

$$\begin{aligned}\sum_{0\le i} p^{q+k} m_{p^i-1} X^{p^i} &= p^{q+k} \log X = p^q \log([p^k]X) \\ &= p^q m_0([p^k]X) + \sum_{0\le i} p^q m_{p^i-1}([p^k]X)^{p^i} \\ &= \sum_{0\le j} p^q a_{k,j} X^{j+1} + \sum_{0\le i} p^q m_{p^i-1}\left(\sum_{0\le j} a_{k,j} X^{j+1}\right)^{p^i}. \end{aligned} \tag{26}$$

Since $s+1$ is not a power of p, when we equate coefficients of X^{s+1} in (21) we get

$$a_{k,s} = -\sum_{U,0<i} m_{p^i-1}\binom{p^i}{U} a_k^U, \tag{27}$$

where the sum is taken over all sequences of nonnegative integers $U = (u_0, u_1, \ldots)$ satisfying (22) and (23). Our strategy is to show that if the sequence U does not satisfy (24) and (25) also, then the terms $m_{p^i-1}\binom{p^i}{U} a_k^U$ of (27) are in I_∞^{g+1} and so can be neglected. Lemma 3.2 of [6] proves that $\binom{p^i}{U} a_k^U \in I_\infty^g$ by a series of inequalities. We shall rework this proof with special attention to the not uncommon conditions forcing $\binom{p^i}{U} a_k^U \in I_\infty^{g+1}$.

Lemma 14. *$(p-1)v_p\binom{p^i}{U} - p^i + \sum_t [u_t - \alpha_p(u_t)](k(p-1)+1)\alpha_p(t+1) \geq -1$ with equality holding only when* (24) *is satisfied.*

Proof.

$$(p-1)v_p\binom{p^i}{U} - p^i + \sum_t [u_t - \alpha_p(u_t)](k(p-1)+1)\alpha_p(t+1) \tag{28}$$

$$= p^i - 1 - \sum_t (p-1)v_p(u_t!) - p^i$$

$$+ \sum_t (p-1)(k(p-1)+1)v_p(u_t!)\alpha_p(t+1) \tag{29}$$

(since $v_p\binom{p^i}{U} = v_p(p^i!) - \sum_t v_p(u_t!) = (p^i-1)/(p-1) - \sum_t v_p(u_t!)$ and by Lemma 3.1(iii) of [3])

$$= -1 + \sum_t (p-1)v_p(u_t!)((k(p-1)+1)\alpha_p(t+1) - 1). \tag{30}$$

Since $(k(p-1)+1)\alpha_p(t+1) - 1 \geq 1(1(2-1)+1) \geq 1$, (30) is larger than -1 unless each $v_p(u_t!) = 0$. □

Lemma 15. *Let $U = (u_0, u_1, \ldots)$ satisfy* (22) *and* (23) *and let g_t be defined for each t by $(p-1)g_t = (k(p-1)+1)\alpha_p(t+1) - 1$. Then $v_p\binom{p^i}{U} + \sum_t u_t g_t \geq g$ with equality holding only when the sequence U satisfies* (24) *and* (25).

Proof. Compute as follows:

$$(p-1)v_p\binom{p^i}{U} + \sum_t (p-1)u_t g_t \tag{31}$$

$$= (p-1)v_p\binom{p^i}{U} + \sum_t u_t(k(p-1)+1)\alpha_p(t+1) - \sum_t u_t \tag{32}$$

$$= (p-1)v_p\binom{p^i}{U} - p^i + \sum_t [u_t - \alpha_p(u_t)](k(p-1)+1)\alpha_p(t+1)$$

$$+ \sum_t (k(p-1)+1)[\alpha_p(u_t)][\alpha_p(t+1)]$$

(by (23) and by arithmetic)

$$\geq \sum_t (k(p-1)+1)[\alpha_p(u_t)][\alpha_p(t+1)] - 1 \tag{33}$$

(with equality if U satisfies (24) by Lemma 14)

$$\geq \sum_t (k(p-1)+1)\alpha_p(u_t(t+1)) - 1$$

$$\geq (k(p-1)+1)\alpha_p(s+1) - 1 = g, \tag{34}$$

with equality holding only when (25) is satisfied by Lemma 9. □

Let

$$x = a_{k,s} + \sum_{U,0\le i} m_{p^i-1}\binom{p^i}{U} a_k^U, \tag{35}$$

where U satisfies (22)–(25). By (16), $p^q m_{p^i-1} \in I_\infty^q$. We have just shown that $p^q x$ is a sum of $m_{p^i-1}\binom{p^i}{U} a_k^U$ each of which is in $I_\infty^q I_\infty^{g+1} = I_\infty^{p+q+1}$. By Lemma 2.2 of [3], $x \in I_\infty^{g+1}$ as required to prove Proposition 12. □

Proposition 16. *Let $s+1$ be a positive integer which is not a power of the prime p. Suppose s is a multiple of $p-1$ and that $\alpha = \alpha_p(s+1)$. Suppose $k \ge 1$ is fixed and that g is defined by $(p-1)g = (k(p-1)+1)\alpha - 1$. Then $a_{k,s}$ satisfies* (36) *and* (37):

$$a_{k,s} \in (p^{(k-1)\alpha}) \cap I_\infty^g \quad + \quad I_\infty^{g+1}, \tag{36}$$

$$a_{k,s} \equiv -\sum_U m_{p-1}\binom{p}{U} a_k^U \quad \text{modulo } (p^{(k-1)\alpha+p}) \cap I_\infty^g + I_\infty^{g+1}, \tag{37}$$

where the sum is taken over all sequences U satisfying (22), (23) (*for $i=1$*), (24), *and* (25).

Proof. By Lemma 11, (36) holds even when $s+1$ is a power of p. So we assume that (36) holds for $a_{k,t}$ where $\alpha_p(t+1) < \alpha_p(s+1)$. Let g_t be defined as in Lemma 15. So we assume each $a_{k,t}$ has the form

$$a_{k,t} = b_t + c_t, \quad b_t \in (p^{(k-1)\alpha_p(t+1)}) \cap I_\infty^{g_t}, \quad c_t \in I_\infty^{g_t+1}. \tag{38}$$

Modulo I_∞^{g+1}, Proposition 12 tells us that

$$a_{k,s} \equiv -\sum_{U,0\le i} m_{p^i-1}\binom{p^i}{U}(b_0+c_0)^{u_0}(b_1+c_1)^{u_1}\dots(b_t+c_t)^{u_t}\dots, \tag{39}$$

where the sequences U satisfy (22)–(25). Since each U satisfies (24), each term of (39) is in BP_* and is in fact in I_∞^g. Any term of (39) having even a single c_t factor will be pushed into I_∞^{g+1}. Thus modulo I_∞^{g+1},

$$a_{k,s} \equiv -\sum_{U,0\le i} m_{p^i-1}\binom{p}{U} b^U, \quad \text{where} \quad b^U = b_0^{u_0} b_1^{u_1} \dots b_t^{u_t} \dots \tag{40}$$

Because the sequence U satisfies (24) and (25), Lemma 9 implies that

$$\sum_t u_t(k-1)\alpha_p(t+1) = (k-1)\alpha_p(s+1). \tag{41}$$

Thus each $b_t \in (p^{(k-1)\alpha_p(t+1)})$ implies that

$$b^U \in (p^{(k-1)\alpha_p(s+1)}). \tag{42}$$

Since U satisfies (24), $v_p\binom{p^i}{U} = v_p(p^i!) = p^{i-1} + \dots + p + 1$. If $i > 2$, $v_p\binom{p^i}{U} \ge p+i$ and thus $m_{p^i-1}\binom{p^i}{U} \in (p^p)$ and $m_{p^i-1}\binom{p^i}{U} b^U \in (p^{(k-1)\alpha+p})$.

The single case remaining to complete the proof is when $i = 2$. By (17), $\binom{p^2}{U} m_{p^2-1} = d p^{p+1} m_{p^2-1} = d p^{p-1}(pv_2 + v_1^{p+1})$ where d is an integer prime to p. So $\binom{p^2}{U} m_{p^2-1} b^U = d p^p v_2 b^U + d p^{p-1} v_1^{p+1} b^U$ with $d p^p v_2 b^U \in (p^{(k-1)\alpha+p})$ and $d p^{p-1} v_1^{p+1} b^U \in I_\infty^{g+1}$. This completes the proof of (37) . The inductive proof of (36) was given in (40) and (42). □

3. Counting monomials

We wish to derive Theorem 1 from Proposition 16. When $p = 2$, Proposition 16 has a merciful simplification. The sum is over sequences $U = (u_0, u_1, \ldots, u_t, \ldots)$ which have $0 \le u_t < 2$ and which add to give the sum 2. Thus for exactly two indices $t = h$ and i, $u_t = 1$. The binomial coefficient $\binom{p}{U}$ is $\binom{2}{1,1} = 2$ and $m_{p-1}\binom{p}{U}$ is $2m_1 = v_1$. The term $g + 1$ is 2α where $\alpha = \alpha_2(s + 1)$ the number of 1's in the dyadic expansion of $s + 1$.

Proposition 17. *Let $s+1$ be a positive integer not a power of 2 and let $\alpha = \alpha_2(s+1)$. Then*

$$a_{k,s} \equiv \sum v_1 a_h a_i \qquad \text{modulo } (2^{(k-1)\alpha+2}) + I_\infty^{2\alpha}, \tag{43}$$

where h and i are nonnegative integers which satisfy (25).

Proof. Since $a_{k,s} \in I_\infty^{2\alpha-1}$, $a_{k,s} - (-a_{k,s}) \in 2I_\infty^{2\alpha-1} \subset I_\infty^{2\alpha}$. For this reason, we can omit the negative sign that was present in (37). □

By Lemma 11, $a_{k,2^n-1} \equiv 2^{n-1} a_{1,2^n-1}$ modulo I_∞^{k+1}. It is well known ([8]) that modulo I_∞^2, $a_{1,2^n-1}$ can be taken as the Brown–Peterson coefficient generator v_n. Thus Theorem 1 holds for those s of the form $s = 2^n - 1$, i.e., those s with $\alpha_2(s + 1) = 1$. We now assume Theorem 1 holds for $a_{k,h}$ and $a_{k,i}$ where $\alpha_2(h + 1)$ and $\alpha_2(i + 1)$ are each less than $\alpha_2(s + 1)$. Modulo $(2^{(k-1)\alpha_2(h+1)+2}) \cap I_\infty^{(k+1)\alpha_2(h+1)-1} + I_\infty^{(k+1)\alpha_2(h+1)}$,

$$a_{k,h} \equiv 2^{(k-1)\alpha_2(h+1)} v_1^{\alpha_2(h+1)-1} v_1^{e'_1} \ldots v_{t-1}^{e'-t} \ldots \tag{44}$$

and modulo $(2^{(k-1)\alpha_2(i+1)+2}) \cap I_\infty^{(k+1)\alpha_2(i+1)-1} + I_\infty^{(k+1)\alpha_2(i+1)}$,

$$a_{k,i} \equiv 2^{(k-1)\alpha_2(i+1)} v_1^{\alpha_2(i+1)-1} v_1^{e''_1} \ldots v_{t-1}^{e''-t} \ldots, \tag{45}$$

where $h = \sum_{0<t} e'_t 2^{t_1}$ and $i = \sum_{0<t} e''_t 2^{t_1}$ are the dyadic expansions of h and i, respectively. By (25) and Lemma 9, for every $t > 0$, $e'_t + e''_t = e_t$ and $\alpha_2(h + 1) + \alpha_2(i+1) = \alpha_2(s+1)$. So each and every $v_1 a_h a_i$ of (43) is a $a_{k,s} = 2^{(k-1)\alpha} v_1^{\alpha-1} 2^{e_1} v_1^{e_2} \ldots v_{t-1}^{e_t}$ as in (3). The indeterminacies match nicely. All that remains to complete the proof of Theorem 1 is to count the number of possible $v_1 a_h a_i$; we want that number to be odd.

We now recall the following cultural facts from [5]. The symbol $\left\{ {n \atop 2} \right\}$ stands for the number of ways n objects can be partitioned into two nonempty and nonordered subsets. This $\{{n \atop 2}\}$ is an example of a Stirling number of the second kind. It is easy to compute that $\{{n \atop 2}\} = 2^{n-1} - 1$. Now recognize that the number of ways to write $s+1 = h+1+i+1$ with the sum subordinate to the dyadic expansion of $s+1$ (25) is just the number of ways to partition the powers of 2 in the dyadic expansion of $s+1$ into two nonempty sets: one set forming the dyadic expansion of $h+1$ and the other forming the dyadic expansion of $i+1$. Thus the number of possible $v_1 a_h a_i$ in (43) is exactly $2^{\alpha_2(s+1)-1} - 1$. In this inductive step, $\alpha_2(s+1) > 1$ and so our number is indeed odd. This completes the proof of Theorem 1. □

What if the prime p is odd? When $p = 3$, there are two ways for an integer to be nonzero modulo (3) and two possibilities for $\binom{p}{U}$: $\binom{3}{2,1}$ and $\binom{3}{1,1,1}$. The necessary combinatorics cannot be handled in a paragraph of cultural notes. When $p = 5$, there are six possibilities for $\binom{5}{U}$ and apparently the combinatorics are even more frightful. We say "apparently" for we have not worked out the analog for Theorem 1 when $p > 3$. The following is the $p = 3$ analog whose proof is unpublished in [7].

Analog 18. *Let* $[3^k]X = \sum_{0 \le s} a_{k,s} X^{s+1}$ *be the* $[3^k]$*-series for Brown–Peterson homology for* $p = 3$. *Let* s *be even and* $s+1$ *have 3-adic expansion* $s+1 = e_1 + e_2 3 + e_3 3^2 + \ldots + e_t 3^{t-1}$, $0 \le e_j < 3$. *Let* d *be the number of* e_j*'s which are 2. Let the integers* f *and* g *be defined by* $2f = \alpha_3(s+1) - 1$ *and* $2g = (2k+1)\alpha_3(s+1) - 1$, *respectively. Then modulo* $(3^{(k-1)\alpha_3(s+1)+3}) \cap I_\infty^g + I_\infty^{g+1}$,

$$a_{k,s} \equiv 2^{f-d}(-1)^f 3^{(k-1)\alpha_3(s+1)} v_1^f 3^{e_1} v_1^{e-2} v_2^{e_3} \ldots v_{t-1}^{e-t}. \tag{46}$$

4. Examples

We conclude with some examples, with **shortest monomials** bold faced. The examples $a_{1,s}$ are taken from Giambalvo's tables [1]. The others are computer calculations made by the second author.

$$a_{1,10} = 31012 v_1 v_2^3 + 5616 v_2 v_3 + \mathbf{17770 v_1^3 v_3} + 161034 v_1^4 v_2^2 + 18997 v_1^7 v_2 + 65744 v_1^{10}.$$

$$a_{1,12} \equiv \mathbf{2 v_1^2 v_2 v_3} + 2 v_1^3 v_2^3 + 6 v_1^5 v_3 + 6 v_1^9 + 4 v_1^{12} + 4 v_2^4 \text{ modulo } (8).$$

$$a_{1,13} \equiv 6 v_1 v_2^4 + \mathbf{7 v_1^3 v_2 v_3} + 5 v_1^4 v_2^3 + 2 v_1^6 v_3 + 4 v_1^7 v_2^2 + 6 v_1^{10} v_2 \text{ modulo } (8).$$

$$a_{2,10} \equiv 24 v_1^{10} + 24 v_1^7 v_2 + 8 v_1^4 v_2^2 + \mathbf{16 v_1^3 v_3} \text{ modulo } (2^5).$$

$$a_{2,11} \equiv 12 v_1^{11} + 3 v_1^8 v_2 + 5 v_1^5 v_2^2 + 2 v_1^2 v_2^3 + 10 v_1^4 v_3 + \mathbf{12 v_1 v_2 v_3} \text{ modulo } (2^4).$$

$$a_{2,13} \equiv 28 v_1^{13} + 26 v_1^{10} v_2 + 10 v_1^7 v_2^2 + 4 v_1^4 v_2^3 + 4 v_1^6 v_3 + \mathbf{24 v_1^3 v_2 v_3} \text{ modulo } (2^5).$$

$$a_{3,10} \equiv 32 v_1^{10} + 192 v_1^7 v_2 + 192 v_1^4 v_2^2 + \mathbf{128 v_1^3 v_3} \text{ modulo } (2^8).$$

$$a_{3,11} \equiv 10 v_1^{11} + 20 v_1^5 v_2^2 + 24 v_1^2 v_2^3 + 24 v_1^4 v_3 + \mathbf{48 v_1 v_2 v_3} \text{ modulo } (2^6).$$

References

[1] V. Giambalvo, Some tables for formal groups and BP, in: Geometric Applications of Homotopy Theory, II, Lecture Notes in Mathematics, Vol. 658 (Springer, Berlin, 1978) 169–176.

[2] M. Hazewinkel, A universal formal group and complex cobordism, Bull. Amer. Math. Soc. 81 (1975) 930–936.

[3] D.C. Johnson, On the $[p]$-series in Brown–Peterson homology, J. Pure Appl. Algebra 48 (1987) 263–270.

[4] D.C. Johnson, The Brown–Peterson [2]-series revisited, preprint, University of Kentucky, 1993.

[5] D.E. Knuth, Two notes on notation, Amer. Math. Monthly 99 (1992) 403–422.

[6] G. Nakos, On the $[p^k]$-series in Brown–Peterson homology, J. Pure Appl. Algebra 66 (1990) 303–309.

[7] G. Nakos, The short monomial for the $[2^k]$- and $[3^k]$-series, preprint, The Johns Hopkins University, 1994.

[8] W.S. Wilson, Brown–Peterson Homology: An Introduction and Sampler, Regional Conf. Ser. in Math., Vol. 48 (AMS, Providence, RI, 1982).